1 Welfare Comparison: Markets *vs.* Quotas

Farmers could only buy large, discrete units of water. We now present a two-period model that illustrates the relevance of this feature. When seasonality, storability, and liquidity constraints are present, and farmers can only buy discrete units of water, the model predicts that the purchase timing differs between constrained and unconstrained farmers, even with perfect information and absent stochastic shocks.

The logic for different purchase timing is that watering before and during the critical season are imperfect substitutes. A unit of water during the critical season conveys more *effective water* than a unit of water outside the critical season. A unit of water during the critical season is therefore more valuable for everyone and has higher price under the market. A poor farmer might not be able to purchase one unit of water during the critical season due to liquidity constraints. The poor farmer might buy a unit of cheaper water prior to the critical season. If units are continuous, the model predicts that wealthy farmers will buy more effective water than poor farmers overall, but not in any particular season. In other words, if farmers could buy a continuous amount of water, a model with perfect information predicts the amount but not timing of water purchased by wealthy and poor farmers.

1.1 Two-Period Model

We present a simple model of water allocation with liquidity-constrained farmers. When farmers are heterogeneous in their productivity, and some farmers are liquidity constrained, the efficiency of markets relative to quotas for welfare is ambiguous. In the absence of liquidity constraints, a market achieves the first-best (FB) allocation. At the other extreme, in the absence of heterogeneity in farmers' productivity, a fixed quota system achieves the FB allocation.

The economy consists of a continuum of a unit mass farmers and one Walrasian auctioneer. There are two goods in the economy: water (x) measured in *liters* and money (μ) measured in *pesetas*. There are two periods denoted by t. For ease of exposition we assume no discounting between periods. The water supply in the economy is constant and equals X_t in period t. In the first period (off season), there is one market to sell X_1 units of water. In the second period (critical season), there is one market to sell X_2 units of water.

Farmers only receive utility from water consumed in the second period. Water purchased in the first period can be used in the second period, but it evaporates at a rate $\delta \in (0, 1)$. This means that for every unit of water purchased in the first period, the farmer will only consume $(1 - \delta)$ units in the second period. In the second period, the total amount of water consumed in the economy equals $X \equiv (1 - \delta) X_1 + X_2$. Farmers differ in their productivity and wealth. First, a productivity type θ_j can take two values $\theta_j \in \{\theta_L, \theta_H\}$ with $\theta_L < \theta_H$, $Pr(\theta_j = \theta_L) = f_L$ and $Pr(\theta_j = \theta_H) = f_H = 1 - f_L$. Second, a wealth type μ_i can take two values $\mu_i = \{\mu_L, \mu_H\}$ with $\mu_L < \mu_H$, $Pr(\mu_i = \mu_L) = g_L$ and $Pr(\mu_i = \mu_H) = g_H = 1 - g_L$. Both θ_j and μ_i are private information.

Farmers' preferences are represented by $u(x_1, x_2, p_1, p_2; \mu_i; \theta_j) = h((1 - \delta)x_1 + x_2; \theta_j) + (\mu_i - p_1 x_1 - p_2 x_2)$. In this equation, $h(x; \theta)$ is a production function that is twice continuously differentiable, strictly increasing in each argument and concave in its first argument, x. The scalar p_t represents the transfer per unit of water received whenever $(\mu_i - p_1 x_1 - p_2 x_2) \geq 0$. In this two-period model, high productivity could mean either a higher productivity shock or a lower soil moisture level. Hence, we can interpret high-productivity farmers as those who have not irrigated in the recent past.

We focus on the case where wealthy farmers are always unconstrained, *i.e.*, $\mu_H \to \infty$. We also focus on non-trivial cases where $\mu_L < h(1; \theta_H)$ such that the wealth of poor but high-productivity farmers is lower than the value they assign to water in the second period. To simplify the exposition, we assume that the realization of θ_j is independent of the realization of μ_i. The independence of θ_j and μ_i does not affect the results qualitatively unless they are perfectly correlated. The welfare analysis is affected by the correlation between productivity and wealth. The greater the correlation, the more efficient the market mechanism.

We restrict attention to the case in which water is scarce, *i.e.*, $X_1 + X_2 < 1$. This assumption implies that, in the unconstrained case, farmers buy at most one unit of water in any period. Rather than solving all possible cases, which are qualitatively similar, we make the following assumptions on the parameter values for a particular case:

*A*1: $g_H (1 + f_H) > X_1 + 2X_2$.

$A2$: $X_2 < g_H < X_1 + X_2$. There are not enough units in the second period for all the wealthy farmers but there are enough units overall for all the wealthy farmers.

$A3$: $X_2 < f_H < X_1 + X_2$. There are not enough units in the second period for all the high-productivity farmers but there are enough units overall for all the high-productivity farmers.

$A4$: $X_2 < g_H f_H < X_1 + X_2$. There are not enough units in the second period for all the wealthy and high-productivity farmers but there are enough units overall for all the wealthy and high-productivity farmers.

Assumption A1 is not needed for the analysis but it allows us to focus on the case where farmers buy zero or one unit. If it is not satisfied, farmers could get multiple units, but the results would be the same. Assumptions A2-A4 allow us to restrict attention to non-trivial cases. We normalize $h(x;\theta)$ so that $h(0;\theta) = 0$ and also require its cross derivative to be positive, *i.e.*, $h_{x\theta}(x;\theta) > 0$. We add another assumption about the concavity of $h(x;\theta)$:

$A5$: $h(1-\delta;\theta_L) > [h(2(1-\delta);\theta_H) - h(1-\delta;\theta_H)]$.

Assumption $A5$ implies that in the first period, the first unit of water for low-productivity farmers is more productive than the second unit of water for high-productivity farmers. This assumption means that the production function $h(x;\theta)$ is *very concave* given θ_L and θ_H. All the assumptions combined guarantee that in any equilibrium no farmer will buy more than one unit in each period and no more than two units in total. Relaxing these assumptions would not change interpretation of the model, but would require the analysis of a large number of cases making the characterization of the model more cumbersome.

The allocation of water in this economy is characterized by four allocation matrices with $ij \in \{LL, LH, HL, HH\}$:

$$Q_{ij} \equiv \begin{bmatrix} q_{ij00} & q_{ij01} & q_{ij02} \\ q_{ij10} & q_{ij11} & q_{ij12} \\ q_{ij20} & q_{ij21} & q_{ij22} \end{bmatrix},$$

where q_{ij,x_1,x_2} represents the mass of individuals with wealth μ_i and productivity θ_j who buy x_1 units in period 1 and x_2 units in period 2. Each allocation matrix satisfies $\sum_{x_1}\sum_{x_2}(q_{ij,x_1,x_2}) = g_i f_j$.

A *Constrained Equilibrium* is characterized by a price vector $p^* \equiv (p_1^*, p_2^*)$ and a general allocation matrix $Q^* \equiv [Q_{LL}^*; Q_{HL}^*; Q_{LH}^*; Q_{HH}^*]$ that satisfy:

- **Optimality (O)**. When $p = p^*$ each farmer maximizes expected utility, *i.e.*, when $p = p^*$ and for each pair (x_1, x_2) such that $q_{ij,x_1,x_2} > 0$, we have $u(x_1, x_2, p_1, p_2; \mu_i; \theta_j) \geq u\left(x_1^{'}, x_2^{'}, p_1, p_2; \mu_i; \theta_j\right)$ for any other pair $\left(x_1^{'}, x_2^{'}\right)$.

- **Resource Constraint (RC)**. The general allocation matrix Q satisfies the resource constraint in each period, *i.e.*, $\sum_{j=L,H}\sum_{i=L,H}\sum_{x_1}(x_1 \cdot q_{ij,x_1,x_2}) = X_1$ and $\sum_{j=L,H}\sum_{i=L,H}\sum_{x_2}(x_2 \cdot q_{ij,x_1,x_2}) = X_2$.

- **Liquidity Constraint (LC)**. For each $i \in \{L, H\}$, the allocation matrices Q_{ij} satisfy LC, *i.e.*, $(x_1 \cdot p_1 + x_2 \cdot p_2) \leq \mu_i$, $\forall i$.

- **Dynamic Consistency (DC).** For every player of type (θ_j, μ_i), the surplus in the second period cannot be greater than the surplus in the first period.

The last condition implies that the equilibrium allocation is dynamically consistent for all players within a type. Given that there are four types, $\{LL, LH, HL, HH\}$, an equilibrium would prescribe that a proportion of each type play a given strategy. To prevent players within the same type trying to imitate players of the same type with a different equilibrium strategy, we require surplus during the second period to be less than or equal to output during the first period. If, to the contrary, the surplus that some type ij players receive during the second period is greater than that of players in the first period, it would be profitable to imitate the strategies of players with larger surpluses during the second period.

Because $h_{\theta x}(\cdot) > 0$, efficiency requires assortative matching (*i.e.,* high productivity farmers consume water in the second period, units that are larger due to evaporation) and no units are wasted (*i.e.,* no farmer consumes more than one unit and all units are consumed). Because $X_2 < g_H < X_1 + X_2$, in the first-best allocation all farmers consuming in the second period have high productivity and some farmers consuming in the first period have low productivity. Thus, in any equilibrium that obtains the first-best allocation we have: $p_1^{FB} \equiv h(1-\delta;\theta_L)$ and $p_2^{FB} \equiv h(1;\theta_H)$.

1.2 Equilibrium

Dependent on the poor farmer's wealth, there are three possible cases. For simplicity, we focus on the case with $p_2^{FB} > \mu_L > p_1^{FB}$. In this case, liquidity constraints affect water allocation (efficiency) but not prices. If μ_L were lower, then prices could be lower but the allocation would be the same.

Equilibrium Allocation

- $(g_H f_H - X_2)$ wealthy and high-productivity farmers buy one unit in the first period and X_2 wealthy and high-productivity farmers buy one unit in the second period, *i.e.*, $q_{HH10} = (g_H f_H - X_2)$ and $q_{HH01} = X_2$.
- $(g_L f_H)$ poor and high-productivity farmers all buy one unit in the first period, *i.e.*, $q_{LH10} = g_L f_H$.
- $(X_1 + X_2 - f_H)$ low-productivity farmers buy water in the first period and $1-(X_1+X_2)$ low-productivity farmers do not buy any water. Because wealthy and low-productivity farmers and poor and low-productivity farmers are indistinguishable, they are allocated randomly according to their relative mass, *i.e.*: $q_{HL10} = (X_1 + X_2 - f_H)\, g_H$, $q_{HL00} = (1-(X_1+X_2))\, g_H$, $q_{LL10} = (X_1 + X_2 - f_H)\, g_L$, and $q_{LL00} = (1-(X_1+X_2))\, g_L$.
- The price in the first period $p_1^* = p_1^{FB}$ is optimal. A no arbitrage condition means that wealthy farmers' surplus cannot be greater in the second period than the one in the first period, thus $p_2^* = p_2^{FB}$. Both prices are optimal.

Proof:

O: We need to check that this is an equilibrium for each allocation such that $q_{ij,x_1,x_2} > 0$ for each type ij.

Low-Productivity Farmers

- **Wealthy Farmers**
  - $q_{HL10} = (X_1 + X_2 - f_H)\, g_H$ farmers are buying one unit in the first period. Because $A5$ holds, they do not want to buy a second unit of water. Because $p_1^* = p_1^{FB}$, they are indifferent between buying a unit of water in the first period and not buying any water. They participated in the rationing and were lucky to get water.
  - $q_{HL00} = (1-(X_1+X_2))\, g_H$ farmers are not buying any water. Because $A5$ holds, they do not want to buy a second unit of water. Because $p_1^* = p_1^{FB}$, they are indifferent between buying a unit of water in the first period and not buying any water. They participated in the rationing but were unlucky and got no water.
- **Poor Farmers**
  - $q_{LL10} = (X_1 + X_2 - f_H)\, g_L$ farmers are buying one unit in the first period. Because $A5$ holds, they do not want to buy a second unit of water. Because $p_1^* = p_1^{FB}$, they are indifferent between buying a unit of water in the first period and not buying any water. They participated in the rationing and were lucky to get water.
  - $q_{LL00} = (1-(X_1+X_2))\, g_L$ farmers are not buying any water. Because $A5$ holds, they do not want to buy a second unit of water. Because $p_1^* = p_1^{FB}$, they are indifferent between buying a unit of water in the first period and not buying any water. They participated in the rationing but were unlucky and got no water.

High-Productivity Farmers

- **Wealthy Farmers.**

  - $q_{HH10} = (g_H f_H - X_2)$ farmers are buying one unit in the first period. Because $p_1^* - p_1^{FB} = p_2^* - p_2^{FB} = 0$, they are indifferent between buying one unit in the first period, one unit in the second period, or not buying at all.
  - $q_{HH01} = X_2$ farmers are buying one unit in the second period. Because $p_1^* - p_1^{FB} = p_2^* - p_2^{FB} = 0$, they are indifferent between buying one unit in the first period, one unit in the second period, or not buying at all.

- **Poor Farmers.**

  - $q_{LH10} = g_L f_H$ farmers are buying one unit in the first period. Because $p_1^* < p_1^{FB}$ they receive a surplus, and are better off than not buying. Because $p_2^* > \mu_L$ they cannot afford to buy one unit in the second period.

Q: $\sum\limits_{x_1}\sum\limits_{x_2} (q_{HL,x_1,x_2}) = (X_1 + X_2 - f_H)\, g_H + (1 - X_1 - X_2)\, g_H = g_H f_L,$

$\sum\limits_{x_1}\sum\limits_{x_2} (q_{LL,x_1,x_2}) = (X_1 + X_2 - f_H)\, g_L + (1 - X_1 - X_2)\, g_L = g_L f_L,$

$\sum\limits_{x_1}\sum\limits_{x_2} (q_{HH,x_1,x_2}) = g_H f_H - X_2 + X_2 = g_H f_H$ and

$\sum\limits_{x_1}\sum\limits_{x_2} (q_{LH,x_1,x_2}) = g_L f_H.$

RC: $\sum\limits_{i=L,H}\sum\limits_{x_1}\sum\limits_{x_2} (x_1 \cdot q_{i\cdot,x_1,x_2}) = (g_H f_H - X_2) + (g_L f_H) + (X_1 + X_2 - f_H) = X_1$

and $\sum\limits_{i=L,H}\sum\limits_{x_1}\sum\limits_{x_2} (x_2 \cdot q_{i\cdot,x_1,x_2}) = 1 \cdot q_{HH01} = X_2$

LC: Because $p_1^* = p_1^{FB} < \mu_L$, LC is trivially satisfied.

DC: Because $p_1^* - p_1^{FB} = p_2^* - p_2^{FB} = 0$, farmers buying water in the first period obtain the same surplus as farmers in the second period. In this case, competition drives all prices to their first-best level, thus no player receives a positive surplus. □

1.3 Empirical Predictions

Market Equilibrium.

The model predicts that, as long as LC are *binding* (*i.e.,* some farmers do not have enough cash to pay for water in a given period), then purchase timing matters. Poor farmers will not be able to buy water during the second period when water is more valuable. Wealthy farmers will buy all the water during the second period and some of the water during the first. The above results imply that as long as LC are *binding,* poor farmers will not be able to buy water during the critical season. The model has ambiguous predictions regarding whether poor farmers will buy more water than wealthy farmers *off season.*

Market Efficiency.

There are two sources of market inefficiency. If LC are *binding* and farmers are homogeneous in productivity, inefficiency will not reduce welfare. As long as all units of water are allocated and no farmer buys more than two units while others get zero units, the allocation would be efficient. If farmers are heterogeneous in productivity, efficiency requires assortative matching between units of water and farmers. In other words, efficiency requires that farmers with high productivity buy water in the second period (and never not buy water) and farmers with low productivity buy water in the first period (and some not buy any water). If LC are *binding*, farmers with high productivity and low wealth might be unable to buy water in the second period when it is more productive. This is the first source of market inefficiency: inefficiency in the intensive margin, or *mismatching*. Mismatching will happen only when differences in productivity exist among farmers, and will be important if differences are large.[1]

Even if there are no differences in productivity among farmers, there could still be market inefficiency. If LC are *severe*, then the market allocation will be inefficient even if farmers are homogeneous in productivity. If LC are *severe,* poor farmers will not be able to buy water in any period and some wealthy farmers will buy more than one unit of water. Because the production function is concave in units of water purchased, gains from a wealthy farmer's second unit of water are less valuable than gains from a poor farmer's first unit of water. This is the second source of market inefficiency: inefficiency in the extensive margin or *overallocation*. *Overallocation* will happen only when LC are *severe*.

Welfare: Market *vs.* Quotas.

The results above describe the conditions under which the market will be inefficient. However, the results do not compare the relative efficiency of the market mechanism *vs.* the quota mechanism when both are inefficient, *i.e.*, when neither achieves the first-best allocation.

Both quotas and markets could achieve the first-best allocation under some conditions. When farmers are heterogeneous in productivity but LC are not *binding*, then markets are efficient but quotas are not. When LC are not *binding,* then the model is similar to the standard neoclassical model, and markets achieve efficiency by allocating more valuable units to those who value them more. Quotas allocate units uniformly. Because units are discrete, uniformly means randomly. This means that farmers with high productivity and farmers with low productivity have the same probability of receiving one unit in each period. Quotas will produce *mismatching* by allocating units during the *critical season* to farmers with low productivity; however all units will be allocated and no farmer will receive more than one unit. Thus, quotas will never *overallocate* units.

When farmers are homogeneous in productivity, but LC are *severe*, then quotas are efficient but markets are not. When farmers are homogeneous in productivity, there is only one potential source of inefficiency: *overallocation*. Because all farmers have the same valuation for each unit, matching is irrelevant. Any mechanism that allocates all units and at most one unit to each farmer will be efficient. However, markets will not pass this test when LC are *severe*. Markets will allocate all units to wealthy farmers and no units to poor farmers. Some wealthy farmers will receive two units, an *overallocation,* and some high-productivity poor farmers will not receive a unit during the *critical season*, thus generating *mismatching*.

In the intermediate case when farmers are heterogeneous in productivity and LC are *binding*, the relative efficiency of markets and quotas is ambiguous. Both mechanisms suffer from *mismatching* but neither suffer from *overallocation*. Both mechanisms assign units during the *critical season* to farmers with low productivity while some high-productivity farmers do not receive any units. In general, quotas are more efficient than markets when heterogeneity in wealth is relatively large and heterogeneity in productivity is relatively small, while markets are more efficient than quotas when heterogeneity in wealth is relatively small and heterogeneity in productivity is relatively large.

[1] If differences in productivity are large compared to the concavity of the production function in units of water purchased, then gains from the second unit of water of a high-productivity farmer could be greater than gains from the first unit of water of a low-productivity farmer. If there is perfect correlation between wealth and productivity, and differences in productivity are large compared to the concavity of the production function, then the market is efficient.